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LINEAR SYSTEMS WITH A QUADRATIC INTEGRAL†

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It is shown that a linear system of n differential equations with constant coefficients, at least one of whose integrals is a non-degenerate quadratic form, may be reduced to a canonical system of Hamiltonian equations. In particular, n is even and the phase flow preserves the standard measure; if the index of the quadratic integral is odd, the trivial solution is unstable, and so on. For the case $n = 4$ the stability conditions are given a geometrical form. The general results are used to investigate small oscillations of non-holonomic systems, and also the problem of the stability of invariant manifolds of non-linear systems that have Morse functions as integrals.

1. BASIC PROPERTIES

LET

$$x^{\cdot} = Ax, \quad x \in \mathbf{R}^n \tag{1.1}$$

be a system of linear differential equations in n-space. The matrix A is assumed to be non-singular. Equivalent formulation: system (1.1) does not have linear non-constant integrals. Let us assume that Eqs (1.1) have an integral which is a non-degenerate quadratic form

$$H = (Bx, x)/2, \quad B^t = B. \tag{1.2}$$

Theorem 1.
1. n is even,
2. $f(-\lambda) = f(\lambda)$, where $f(\lambda) = |A - \lambda E|$ is the characteristic polynomial of A,
3. $\operatorname{div}(Ax) = \operatorname{tr} A = 0$,
4. if the index of the form (1.2) is odd, the equilibrium $x = 0$ is unstable,
5. system (1.1) has $n/2$ independent quadratic integrals,
6. the equilibrium $x = 0$ is stable if and only if (1.1) has a positive definite quadratic integral.

Indeed, if H is an integral of Eqs (1.1), then

$$H^{\cdot} = (x, BAx) \equiv 0.$$

Consequently, the matrix $D = BA$ is skew-symmetric: $BA = -A^tB$. Since $|D| \neq 0$, it follows that n is even (a skew-symmetric matrix of odd order is always singular). Furthermore,

$$f(\lambda) = |B||A - \lambda E||B^{-1}| = |BA - \lambda B||B^{-1}| = |A^tB + \lambda B||B^{-1}| =$$
$$= |A^t + \lambda E| = f(-\lambda).$$

We have thus proved property 2. Since $\operatorname{tr} A$ is the coefficient of $(-\lambda)^{2n-1}$ in the characteristic polynomial, property 2 implies property 3. In particular, the phase flow of system (1.1) preserves the standard measure in $\mathbf{R}^n$. Since n is even, we have $f(\lambda) \to +\infty$ as $\lambda \to \infty$. As the skew-symmetric

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matrix D is non-singular, $|D|>0$. Since ind$H$ is odd, it follows that $|B|<0$. Consequently, $f(0)=|A|=|B^{-1}||D|<0$. Continuity considerations imply that f has a real positive zero and therefore the equilibrium $x=0$ is indeed unstable. Property 4 was first pointed out in [1]; it extends a classical result due to Kelvin, concerning gyroscopic stabilization conditions, to the general case of linear systems with a quadratic integral. We know of no simple direct proofs of properties 5 and 6 of the theorem; they will be proved in the next section.

Note that property 3 also holds without the assumption that A is non-singular. Indeed,

$$A=B^{-1}D,\ A^t=-DB^{-1}.$$

Consequently,

$$\mathrm{tr}A=\mathrm{tr}A^t=\mathrm{tr}B^{-1}D=\mathrm{tr}DB^{-1}\quad=-\mathrm{tr}DB^{-1}$$

which implies the desired conclusion.

Remark. Let $|A|=0$ and suppose that the characteristic polynomial f has m zero roots with simple elementary divisors. Then system (1.1) will have m independent linear integrals and its restriction to the $(n-m)$-dimensional plane of zero values of these integrals will be a non-degenerate linear system. This system has a quadratic integral (the restriction of G) and Theorem 1 may therefore be applied. If there is a multiple root zero with a non-trivial Jordan block, the equilibrium $x=0$ is unstable.

2. REDUCTION TO CANONICAL FORM

Theorem 1 shows that linear systems with quadratic integrals have many characteristic properties of linear Hamiltonian systems. It turns out that this is no accident.

Define a bilinear form

$$\omega(x',x'')=(\Omega x',x''),\ \Omega=BA^{-1}\tag{2.1}$$

Lemma. $(\omega,\mathbf{R}^n)$ is a symplectic space.

To prove this, we have to check that the form ω is non-degenerate, skew-symmetric and closed ($d\omega=0$). The first and third properties are obvious, so it remains to prove that Ω is a skew-symmetric matrix. Indeed, by Sec. 1, $A^tB=-BA$. Consequently,

$$A^t=-BAB^{-1}\qquad (A^t)^{-1}=-BA^{-1}B^{-1}.$$

Therefore,

$$\Omega^t=(A^t)^{-1}B=-(BA^{-1}B^{-1}),\quad B=-BA^{-1}=-\Omega,$$

as required.

Theorem 2. The linear system (1.1) is Hamiltonian; the symplectic structure is defined by the form (2.1) and the Hamiltonian is the integral (1.2).

Indeed,

$$\omega(x^{\cdot},\ \cdot\ )=(\Omega x^{\cdot},\ \cdot\ )=(Bx,\ \cdot\ )=dH(\ \cdot\ ).$$

We now outline a procedure for reducing system (1.1) to standard Hamiltonian form. Since the skew-symmetric matrix Ω is non-singular, there exists a non-singular matrix C such that

$$C^t\Omega C=C^tBA^{-1}C=-J,$$

where

$$J=\begin{Vmatrix}0 & -E_n\\ E_n & 0\end{Vmatrix}$$

is a unit symplectic matrix. Set $x = Cz$. In the new variables $z^{\bullet} = C^{-1}ACz$, $H = (C^{t}BCz, z)/2$. Consequently, $z^{\bullet} = J\partial H/\partial z$.

The variables z_k and $z_{n/2+k}$ are canonically conjugate.

Theorem 1 is a corollary of Theorem 2 and certain well-known results of Hamiltonian mechanics. For example, property 3 is Liouville's theorem on the conservation of phase volume. Property 4 may be deduced from the fact that the Hamiltonian of a linear system can be reduced in the stable case to the form

$$H = \tfrac{1}{2}\, \Sigma a_i (x_i^2 + y_i^2), \quad a_i \neq 0 \tag{2.2}$$

The index of this quadratic form is even. The system with Hamiltonian (2.2) has a positive definite integral

$$F = \Sigma (x_i^2 + y_i^2)$$

which proves property 6. The proof of property 5 of Theorem 1 follows from Williamson's results on the classification of normal forms of quadratic Hamiltonians [2] (see also [3, Sec. 21]).

3. THE CASE $n = 4$

Let us look more closely at the simplest non-trivial case, when $n = 4$. If the index of the quadratic form H is 0 or 4, the equilibrium position is stable (H is a Lyapunov function). But if the index is 1 or 3, we have unstable equilibrium (property 4 of Theorem 1). When ind$H = 2$, the equilibrium may be either stable or unstable. We will now consider the question of distinguishing these cases, without having to evaluate the characteristic values of the matrix A.

Let $\mathbf{G}_2$ be the four-dimensional Grassmann manifold of all the two-dimensional planes through the origin of $\mathbf{R}^4$ We will call a plane π Lagrangian if $\omega(x', x'') = 0$ for all $x', x'' \in \pi$. The set of all Lagrangian planes forms a three-dimensional submanifold $\Lambda_2 \subset \mathbf{G}_2$.

A quadratic form H of index 2 transforms $\mathbf{R}^4$ into a pseudo-Euclidean space of type (2.2), often called an Artin space. The geometry of Artin spaces has been thoroughly studied (see, e.g. [4]). It turns out that through every straight line on the isotropic cone $x : H(x) = 0$ that contains the origin there pass exactly two two-dimensional planes π_1 and π_2, called totally singular planes. The set of singular planes is the union of two connected one-dimensional components (two regular closed curves in $\mathbf{G}_2$), which we will call singular orbits. What can we say of the positions of these curves relative to the submanifold Λ_2? The answer is given by the following theorem.

Theorem 3. The number of intersections of two singular orbits with Λ_2 is given by Table 1.

The plus sign signifies the existence of a Jordan block, and the minus sign its non-existence. The symbol ∞ means that the orbit in question lies entirely within Λ_2. As can be seen from Table 1, different types of Hamiltonians are associated with different numbers of intersections.

TABLE 1

No.	Eigenvalues $a, b \in \mathbf{R}$	No. of intersections	
		with the first orbit	with the second orbit
1	$\pm ia, \pm ib; a \neq b$	0	0
2	$\pm ia, \pm ia; -$	∞	0
3	$\pm ia, \pm ia; +$	1	0
4	$\pm a, \pm b; a \neq b$	2	2
5	$\pm a, \pm a; -$	∞	2
6	$\pm a, \pm a; +$	2	1
7	$\pm a, \pm ib$	2	0

The proof of Theorem 3 uses Williamson's theory of normal forms [2]. In case 1 the Hamiltonian may be reduced to the form

$$H = a(x_1^2 + y_1^2)/2 - b(x_2^2 + y_2^2)/2$$

where a and b are positive real numbers, and x_s and y_s are conjugate canonical variables. Two different families of totally singular planes exist

$$L_\xi:\ a^{1/2} y_1 = a^{1/2}\operatorname{sh}\xi\, x_1 + b^{1/2}\operatorname{ch}\xi\, x_2$$
$$b^{1/2} y_2 = a^{1/2}\operatorname{ch}\xi\, x_1 + b^{1/2}\operatorname{sh}\xi\, x_2$$
$$N_\eta:\ a^{1/2} y_1 = a^{1/2}\operatorname{sh}\eta\, x_1 + b^{1/2}\operatorname{ch}\eta\, x_2$$
$$b^{1/2} y_2 = -a^{1/2}\operatorname{ch}\eta\, x_1 - b^{1/2}\operatorname{sh}\eta\, x_2$$

Here ξ and η are real parameters; as $\xi, \eta \to \pm\infty$, these planes become

$$L_{\pm\infty}:\ a^{1/2} y_1 = \pm b^{1/2} y_2, \quad a^{1/2} x_1 = \mp b^{1/2} x_2$$
$$N_{\pm\infty}:\ a^{1/2} y_1 = \mp b^{1/2} y_2, \quad a^{1/2} x_1 = \mp b^{1/2} x_2$$

The singular planes of the same family (L or N) intersect only at the origin, but planes from different families always intersect along an isotropic straight line. If $a \neq b$, not one of these planes is Lagrangian (in the standard symplectic structure); the restriction of the 1-form $y_1 dx_1 + y_2 dx_2$ is not a total differential. But if $a = b$ (type 2), all the planes in orbit L are Lagrangian (including $L_{\pm\infty}$), but there are no Lagrangian planes in orbit N.

Now consider type 3. By a well-known result [2], the Hamiltonians of this case reduce to the form

$$H = \pm(a^{-2} x_1^2 + x_2^2) - a^2 y_1 x_2 + y_2 x_1, \quad a \neq 0$$

To fix our ideas, let us take the plus sign before the parentheses. The singular orbits consist of the following planes

$$L_\xi:\ y_1 = \xi x_1 + (2a^2)^{-1} x_2, \quad y_2 = -(2a^2)^{-1} x_1 + a^2 \xi x_2$$
$$N_\eta:\ y_1 = \frac{1 + a^2\eta^2}{2a^4\eta}\, x_1 + \frac{1}{a^2\eta}\, y_2, \quad x_2 = \eta x_1$$

The orbit L has exactly one Lagrangian plane $L_\infty = \{x_1 = x_2 = 0\}$, while the orbit N has none.

Types 4–7 are considered in analogous fashion.

As an illustrative example, we will determine the condition for gyroscopic stabilization of the equilibrium position of the Hamiltonian system

$$\ddot{x}_1 = -\omega \dot{x}_2 + a^2 x_1, \quad \ddot{x}_2 = \omega \dot{x}_1 + b^2 x_2; \qquad a, b > 0$$

This system has a quadratic integral

$$H = \dot{x}_1^2 + \dot{x}_2^2 - a^2 x_1^2 - b^2 x_2^2, \qquad \operatorname{ind} H = 2$$

The equations of the totally singular planes are

$$L_\varphi^\pm:\ \dot{x}_1 = a\cos\varphi\, x_1 + b\sin\varphi\, x_2, \quad \dot{x}_2 = \pm a\sin\varphi\, x_1 \pm b\cos\varphi\, x_2$$

Since $\dot{x}_1 = y_1 - \omega x_2/2$, $\dot{x}_2 = y_2 + \omega x_1/2$, the condition for the plane $L_\varphi^\pm$ to be Lagrangian is

$$\omega = \pm(a \mp b)\sin\varphi$$

Consequently, if $|\omega| > a + b$, none of the singular planes is Lagrangian. By Theorem 3, this condition guarantees stability of the equilibrium position $x_1 = x_2 = 0$.

4. SOME APPLICATIONS

Let us consider a linear system whose dynamics is described by the following second-order differential equations

$$x^{\cdot\cdot} = Ax, \quad x \in \mathbf{R}^n \tag{4.1}$$

where A is a constant matrix (in the general case $A' \neq A$).

We know [5] that the linearized equations of motion of a non-holonomic system in a potential force field can be reduced to the form (4.1) near the equilibrium position. If the equilibrium is not a critical point of the potential energy, then A is generally not a symmetric matrix.

Theorem 4. Equations (4.1) have n independent quadratic integrals

$$(Xx^{\cdot},\ x^{\cdot})/2 - (Yx, x)/2 \tag{4.2}$$

and, if $|A| \neq 0$, there is a non-degenerate integral (4.2) ($|X| \neq 0$, $|Y| \neq 0$).

Corollary 1. If $|A| \neq 0$, Eqs (4.1) are Hamiltonian.

The proof of Theorem 4 uses the following auxiliary result [6]: for any matrix A, symmetric matrices X and Y, $|X| \leq \neq 0$ exist such that

$$XA = Y \tag{4.3}$$

In particular, any matrix may be represented (but not uniquely) as a product of two symmetric matrices. The set of pairs X, Y satisfying (4.3) is a linear space of dimension

$$2\,[n\,(n+1)/2] - n^2 = n$$

The function (4.2) is an integral of system (4.1) if and only if Eq (4.3) is satisfied. If $|A| \neq 0$, then $|Y| \neq 0$. In that case the quadratic form (4.2) is non-degenerate. This proves the theorem.

It should be noted that Eqs (4.1) are equivalent to the Lagrange equations with Lagrangian

$$L = T - V = (Xx^{\cdot}, x^{\cdot})/2 + (Yx, x)/2$$

Since the "kinetic energy" T is not always positive definite, it follows that in the general case system (4.1) does not split into n independent oscillators. When A has n linearly independent vectors with real eigenvalues or the matrix Y is positive (negative) definite, the coordinates $x_1, \ldots, x_n$ are separated.

Corollary 2. Suppose that the "potential energy" V is positive definite. Then the equilibrium $x = 0$ of system (4.1) is stable if and only if the "kinetic energy" T has a strict minimum at $x^{\cdot} = 0$.

As an example, consider the mechanical system with kinetic energy $\frac{1}{2}(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2})$ and force function $z + (ax^2 + by^2)/2$, subject to a non-holonomic constraint $z^{\cdot} = cx^{\cdot}y$ [7]. It is assumed that the constants a, b, c are not zero. This system has a whole family of equilibrium positions

$$x = y = 0, \quad z = \text{const}$$

The linearized equations of motion have the form (4.1) with a non-symmetric matrix A

$$x^{\cdot\cdot} = ax + cy, \quad y^{\cdot\cdot} = by \tag{4.4}$$

The equilibrium is stable if a, $b < 0$ and $a \neq b$.

If $a \neq b$, Eqs (4.4) have two independent quadratic integrals

$$F = y^{\cdot 2} - by^2, \quad \Phi = [(a-b)\,x^{\cdot} + cy^{\cdot}]^2 - a\,[(a-b)\,x + cy]^2$$

If a, $b < 0$, the sum of these integrals is positive definite. This result corresponds to part 6 of Theorem 1.

If $a = b$, the integrals F and Φ are dependent. By property 5 of Theorem 1, in this case a second independent quadratic integral must also exist. It will be a non-degenerate quadratic form

$$\Phi = x^{\cdot}\,y^{\cdot} - axy - cy^2/2$$

5. INVARIANT MANIFOLDS

The results of Secs 1 and 2 are applicable not only to equations linearized in the neighbourhood of equilibrium positions. They may be extended (modulo some additional assumptions) to systems in the neighbourhood of invariant manifolds.

Let M be a compact m-dimensional invariant manifold of a dynamical system whose restriction to M possesses an invariant measure of density $\rho>0$. It is assumed that the system is ergodic on M.

The simplest example is conditionally periodic motion on an m-dimensional torus $M = \mathbf{T}^m$

$$\dot{\varphi_1} = \omega_1, \ldots, \quad \dot{\varphi_m} = \omega_m; \quad \omega_j = \text{const}$$

If the frequencies $\omega_1, \ldots, \omega_m$ are incommensurable, this system is ergodic.

Let x be local coordinates on M and let y be coordinates in the transversal directions. In these variables, the invariant manifold M is defined by the relation $y = 0$, and the differential equations are

$$x^{\cdot} = v(x) + f(x, y), \quad y = \Omega y + g(x, y)$$
$$f(x, 0) = 0, \quad g = O(|y|^2) \tag{5.1}$$

We shall consider what is known as the reduced case, when the matrix Ω is constant in suitable coordinates. A discussion of the reducibility of invarient tori may be found in [8]. When $m = 1$ the equations are always reducible (the Lyapunov–Floquet theorem).

Suppose that in the neighbourhood of M the system has an integral

$$H(x, y) = H_0(x) + (y, h(x)) + (A(x)y, y)/2 + \ldots \tag{5.2}$$

Since $H^{\cdot} \equiv 0$, it follows that

$$(\partial H_0 / \partial x, v) = 0, \quad (\partial h/\partial x, v) = -\Omega^t h$$
$$(\partial A/\partial x\, vy, y)/2 + (A\Omega y, y) = 0, \ldots \tag{5.3}$$

The first relation implies that H_0 is an integral of the dynamical system on M. In view of ergodicity, $H_0 \equiv \text{const}$.

We will now assume that M is non-degenerate: the covectors of the field h on M, which satisfy the second equation of (5.3), vanish identically. In particular, Ω is a non-singular matrix (otherwise the equation would have a non-trivial solution $h = \text{const}$). On a torus with conditionally periodic motion, non-degeneracy means that Ω has no eigenvalues of the form $i(k_1\omega_1 + \ldots + k_m\omega_m)$, $k_j \in \mathbf{Z}$.

Multiply the third equation of (5.3) by ρ and integrate over M. Averaging the first term gives zero, because

$$\int_M \rho\left(\frac{\partial a}{\partial x}, v\right) d^m x = -\int_M a \operatorname{div}(\rho v)\, d^m x = 0$$

Now put

$$A^* = \int_M \rho A\, d^m x$$

Then $(A^*\Omega y, y) = 0$. Consequently, the quadratic form

$$(A^* z, z) \tag{5.4}$$

is an integral of the linear system

$$z^{\cdot} = \Omega z \tag{5.5}$$

If A^* is non-singular, we can apply the results of Secs 1 and 2. In particular the following theorem holds.

Theorem 5. Suppose that the non-degenerate quadratic form (5.4) has an odd index. Then the invariant manifold M is unstable.

The instability of M in the linear approximation follows from part 4 of Theorem 1: at least one eigenvalue of the matrix Ω is positive. The instability of M in the strictly non-linear setting of the problem is proved by using the following observation: the Lyapunov function of the linearized equations (5.5) can also serve as a Lyapunov function for system (5.1).

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